Name ______________________ ID3 _ _ _

**ECS 332: Principles of Communications** **2017/1**

# HW 5 — Due: Nov 3, 4 PM

*Lecturer: Prapun Suksompong, Ph.D.*

## Instructions

(a) This assignment has 8 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1.** Consider a "square" wave (a train of rectangular pulses) shown in Figure 5.1. Its values periodically alternates between two values $A$ and 0 with period $T_0$. At $t = 0$, its value is $A$.

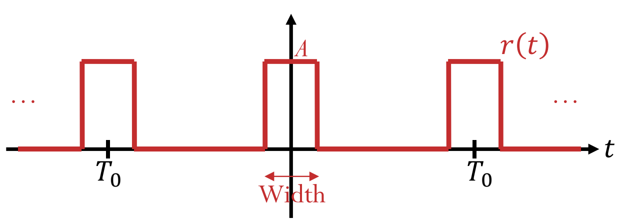


Figure 5.1: A train of rectangular pulses

Some values of its Fourier series coefficients are provided in the table below:

| $k$ | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_k$ | $-\frac{\sqrt{2}}{7\pi}$ | $-\frac{1}{3\pi}$ | $-\frac{\sqrt{2}}{5\pi}$ | 0 | $\frac{\sqrt{2}}{3\pi}$ | $\frac{1}{\pi}$ | $\frac{\sqrt{2}}{\pi}$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{\pi}$ | $\frac{1}{\pi}$ | $\frac{\sqrt{2}}{3\pi}$ | 0 | $-\frac{\sqrt{2}}{5\pi}$ | $-\frac{1}{3\pi}$ | $-\frac{\sqrt{2}}{7\pi}$ |

(a) Find its duty cycle.

In lecture, we found that when duty cycle $= \frac{1}{n}$,
$c_n = 0$ along with $c_k$ for $k$ being multiple(s) of $n$.
Here, $c_4 = 0 \Rightarrow n = 4 \Rightarrow$ duty cycle $= \frac{1}{4}$

(b) Find the value of $A$. (Hint: Use $c_0$.)

**Problem 2.** You are asked to design a DSB-SC modulator to generate a modulated signal $km(t)\cos(2\pi f_c t)$, where $m(t)$ is a signal band-limited to $B$ Hz. Figure 5.2 shows a DSB-SC modulator available in the stockroom. Note that, as usual, $\omega_c = 2\pi f_c$. The carrier generator available generates not $\cos(2\pi f_c t)$, but $\cos^3(2\pi f_c t)$. Explain whether you would be able to generate the desired signal using only this equipment. You may use any kind of filter you like. [Lathi and Ding, 2009, Q4.2-3]

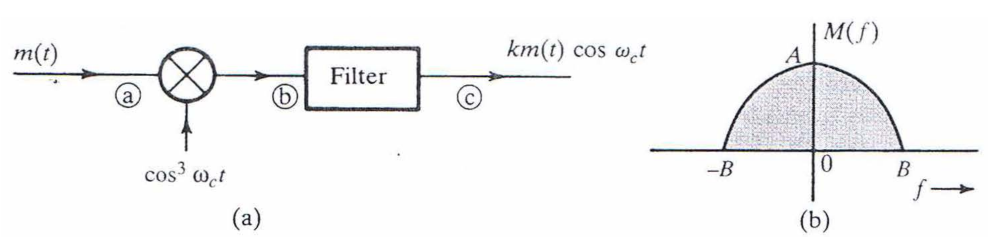


Figure 5.2: Problem 2

(a) We know that a real-valued signal $r(t)$ that is even and periodic with period $T_0$ can be expanded using Fourier series into

$$r(t) = c_0 + a_1\cos(2\pi f_0 t) + a_2\cos(2\pi(2f_0)t) + a_3\cos(2\pi(3f_0)t) + \cdots \quad (5.1)$$

where $f_0 = \frac{1}{T_0}$. Consider the signal $r(t) = \cos^3(2\pi f_c t)$.

(i) Is it periodic?

Yes $\cos(2\pi f_c t)$ is periodic. A function of another periodic function is also periodic.

(ii) Is it even?

Yes $r(-t) = (\cos(-2\pi f_c t))^3 = \cos^3(2\pi f_c t) = r(t)$

$\cos(-x) = \cos(x)$

(iii) Expand $r(t) = \cos^3(2\pi f_c t)$ into a linear combination of $\cos(2\pi(nf_c)t)$ as in (5.1) above.

$$\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$$

$$\cos^2(A) = \frac{1}{2}(1+\cos(2A)) = \frac{1}{2} + \frac{1}{2}\cos(2A)$$

$$\cos^3(A) = (\cos^2(A))(\cos(A)) = \frac{1}{2}\cos A + \frac{1}{2}\cos(2A)\cos(A)$$

$$= \frac{1}{2}\cos A + \frac{1}{4}(\cos(3A) + \cos(A))$$

$$= \frac{3}{4}\cos A + \frac{1}{4}\cos(3A)$$

$$\cos^3(2\pi f_c t) = \frac{3}{4}\cos(2\pi f_c t) + \frac{1}{4}\cos(2\pi(3f_c)t)$$

$f_0 = f_c$
$c_0 = 0$
$a_1 = \frac{3}{4}$
$a_2 = 0$
$a_3 = \frac{1}{4}$
$a_k = 0, k \geq 4$

(b) What kind of filter is required in Figure 5.2?

(c) Determine the signal spectra at points (b) and (c) in Figure 5.2, and indicate the frequency bands occupied by these spectra.

(d) What is the minimum usable value of $f_c$?

(e) Would this scheme work if the carrier generator output were $\cos^2(2\pi f_c t)$? Explain.

**Problem 3.** Consider an AM transmitter.

(a) Suppose the message is $m(t) = 4\cos(10\pi t)$ and the transmitted signal is

$$x_{\text{AM}}(t) = A\cos(100\pi t) + m(t)\cos(100\pi t).$$

Find the value of $A$ which yields the modulation index in each part below.

(i) $\mu = 50\%$

(ii) $\mu = 100\%$

(iii) $\mu = 150\%$

(b) Suppose the message is $m(t) = \alpha\cos(10\pi t)$ and the transmitted signal is

$$x_{\text{AM}}(t) = 4\cos(100\pi t) + m(t)\cos(100\pi t).$$

Find the value of $\alpha$ which yields the modulation index in each part below.

(i) $\mu = 50\%$

(ii) $\mu = 100\%$

(iii) $\mu = 150\%$

# Extra Questions

Here are some optional questions for those who want more practice.

**Problem 4** (M2011Q5)**.** In this question, you are provided with a partial proof of an important result in the study of Fourier transform. Your task is to figure out the quantities/-expressions inside the boxes labeled a,b,c, and d.

We start with a function $g(t)$. Then, we define $x(t) = \sum_{\ell=-\infty}^{\infty} g(t - \ell T)$. It is a sum that involves $g(t)$. What you will see next is our attempt to find another expression for $x(t)$ in terms of a sum that involves $G(f)$.

To do this, we first write $x(t)$ as $x(t) = g(t) * \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T)$. Then, by the convolution-in-time property, we know that $X(f)$ is given by

$$X(f) = G(f) \times \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right)$$

We can get $x(t)$ back from $X(f)$ by the inverse Fourier transform formula: $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$. After plugging in the expression for $X(f)$ from above, we get

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} e^{j2\pi ft} G(f) \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right) df \\ &= \boxed{\quad a \quad} \int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} e^{j2\pi ft} G(f) \delta\left(f + \boxed{\qquad b \qquad}\right) df. \end{aligned}$$

By interchanging the order of summation and integration, we have

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi ft} G(f) \delta\left(f + \boxed{\qquad b \qquad}\right) df.$$

We can now evaluate the integral via the sifting property of the delta function and get

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} e^{\boxed{\quad c \quad}} G\left(\boxed{\quad d \quad}\right).$$

**Problem 5.** Would the scheme in Problem 2 work if the carrier generator output were $\cos^n \omega_c t$ for any integer $n \geq 2$?

**Problem 6.** Consider the basic DSB-SC transceiver with time-delay channel presented in class. Recall that the input of the receiver is

$$x(t - \tau) = m(t - \tau)\sqrt{2}\cos(\omega_c(t - \tau))$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to $B$, i.e., $|M(f)| = 0$ for $|f| > B$. We also assume that $f_c \gg B$.

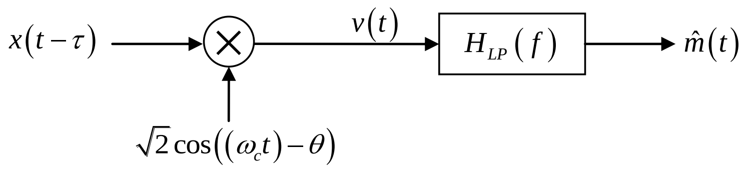


Figure 5.3: Receiver for Problem 6a

(a) Suppose that, at the receiver, we multiply by $\sqrt{2}\cos((\omega_c t) - \theta)$ instead of $\sqrt{2}\cos(\omega_c t)$ as illustrated in Figure 5.3. Assume

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

Find $\hat{m}(t)$ (the output of the LPF).

(b) Use the same assumptions as part (a). However, at the receiver, instead of multiplying by $\sqrt{2}\cos((\omega_c t) - \theta)$, we pass $x(t-\tau)$ through a half-wave rectifier (HWR) as shown in Figure 5.4.

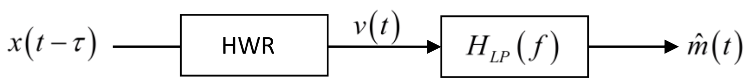


Figure 5.4: Receiver for Problem 6b

Make an extra assumption that $m(t) \ge 0$ for all time $t$ and that the half-wave rectifier input-output relation is described by a function $f(\cdot)$:

$$f(x) = \begin{cases} x, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Find $\hat{m}(t)$ (the output of the LPF).

**Problem 7** (M2011Q7)**.** Suppose $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to $W$, i.e., $|M(f)| = 0$ for $|f| > W$. Consider a DSB-SC transceiver shown in Figure 5.5.

Also assume that $f_c \gg W$ and that $H_{LP}(f) = \begin{cases} 1, & |f| \le W \\ 0, & \text{otherwise.} \end{cases}$

Make an extra assumption that $m(t) \ge 0$ for all time $t$ and that the full-wave rectifier (FWR) input-output relation is described by a function $f_{FWR}(\cdot)$:

$$f_{FWR}(x) = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

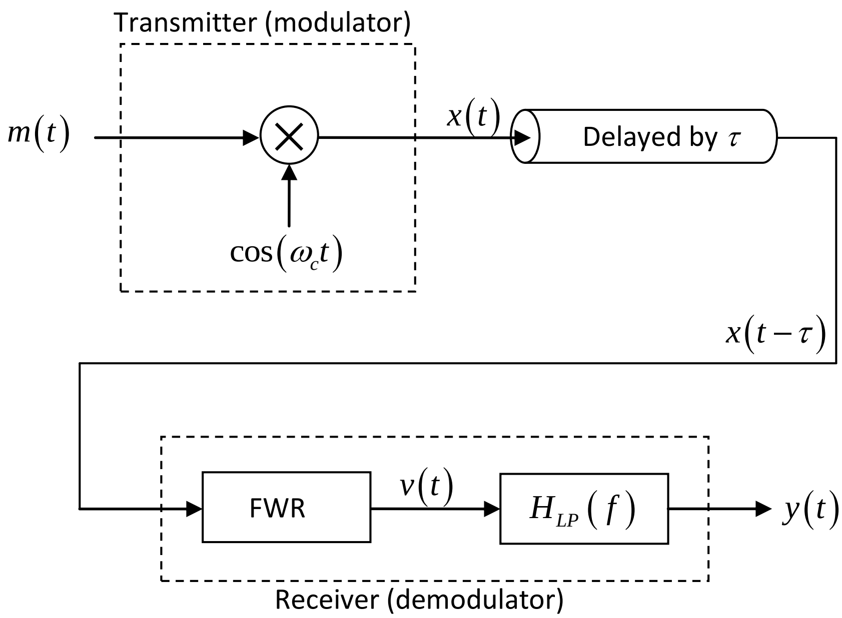


Figure 5.5: A DSB-SC transceiver

(a) Recall that the **half**-wave rectifier input-output relation is described by a function $f_{HWR}(\cdot)$ : $f_{HWR}(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$ We have seen in Problem 6b that when the receiver uses **half**-wave rectifier,

$$v(t) = x(t-\tau) \times g_{HWR}(t-\tau)$$

where $g_{HWR}(t) = 1[\cos(\omega_c t) \geq 0]$.

(i) The receiver in this question uses **full**-wave rectifier. Its $v(t)$ can be described in a similar manner; that is

$$v(t) = x(t-\tau) \times g_{FWR}(t-\tau).$$

Find $g_{FWR}(t)$. Hint: $g_{FWR}(t) = c_1 \times g_{HWR}(t) + c_2$ for some constants $c_1$ and $c_2$. Find these constants.

(ii) Recall that the Fourier series expansion of $g_{HWR}(t)$ is given by

$$g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \frac{1}{7}\cos 7\omega_c t + \dots\right).$$

Find the Fourier series expansion of $g_{FWR}(t)$.

(b) Find $y(t)$ (the output of the LPF).